Improving FFT Frequency Measurement Resolution by Parabolic and Gaussian Spectrum Interpolation

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Abstract. Discrete spectra can be used to measure frequencies of sinusoidal signal components. Such a measurement consists in digitizing a compound signal, performing windowing of the signal samples and computing their discrete magnitude spectrum, usually by means of the Fast Fourier Transform algorithm. Frequencies of individual components can be evaluated from their locations in the discrete spectrum with a resolution depending on the number of samples. However, the frequency of a sinusoidal component can be determined with improved resolution by fitting an interpolating parabola through the three largest consecutive spectrum bins corresponding to the component. The abscissa of its maximum constitutes a better frequency approximation. Such a method has been used for tune measurement systems in circular accelerators. This paper describes the efficiency of the method, depending on the windowing function applied to the signal samples. A typical interpolation gain is one order of magnitude. Better results are obtained with Gaussian interpolation, offering frequency resolution improvement by more than two orders of magnitude when used with windows having fast sidelobe decay. An improvement beyond three orders of magnitude is possible with steep Gaussian windows. These results are confirmed by laboratory measurements. Both methods assume the measured frequency to be constant during acquisition and the spectral peak corresponding to the measured component to constitute a local maximum in a given band of the input signal discrete spectrum.

Window	Main lobe -6dB width [bin]	Highest sidelobe [dB]	Sidelobe fall-off [dB/oct]	Parabolic interpolation E_{max} [% of D_f]	Parabolic interpolation $\lvert j_{in} - k_m \rvert$ @ E_{max}	Parabolic interpolation Gain	Gaussian interpolation E_{max} [% of D_f]	Gaussian interpolation $\lvert j_{in} - k_m \rvert$ @ E_{max}	Gaussian interpolation Gain
Hanning	2.00	-31.5	18	5.28	0.307	**9.5**	1.60	0.291	**31.2**
Blackman	2.30	-58.1	18	4.38	0.303	**11.4**	0.66	0.289	**75.3**
3T1	2.36	-64.2	18	4.18	0.303	**11.9**	0.59	0.289	**84.7**
3T3	2.59	-46.7	30	3.40	0.300	**14.7**	0.53	0.289	**93.7**
4T1	2.69	-93.3	18	3.34	0.300	**15.0**	0.31	0.289	**159**
4T3	2.83	-82.6	30	2.99	0.299	**16.7**	0.31	0.289	**163**
4T5	3.07	-60.9	42	2.51	0.297	**19.9**	0.27	0.289	**187**
Gaussian $r=6$	2.26	-56.1	6	4.95	0.305	**10.1**	0.24	0.281	**208**
Gaussian $r=7$	2.62	-71.0	6	3.80	0.301	**13.2**	0.052	0.279	**970**
Gaussian $r=8$	3.00	-87.6	6	2.95	0.298	**17.0**	0.0087	0.278	**5756**

FFT Frequency Measurement

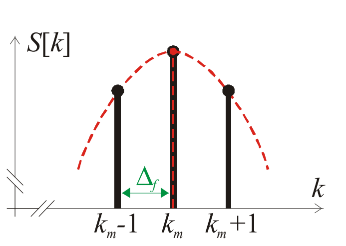
Assume that a compound signal $s(t)$ has been sampled with frequency f_s and contains a sinusoidal component $s_{in}(t)$, whose frequency f_{in} is to be measured. The discrete magnitude spectrum $S[k]$ of the signal sample sequence $s[n]=s(nT_s)$ is calculated at frequencies that are integer multiples of $D_f=f_s/N$. If $S[k]$ has an observable local maximum corresponding to $s_{in}(t)$ at the spectrum bin k_m, f_{in} can be approximated as

$$f_{in} \cong k_m \Delta_f = k_m \frac{f_s}{N} = \frac{k_m}{N T_s} = \frac{k_m}{L} \qquad (1)$$

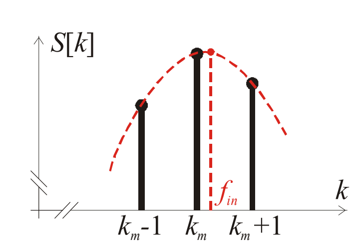
where $T_s=f_s^{-1}$ is the sampling period, N is the total number of samples and $L=NT_s$ is the sampling duration. The largest approximation error $e=\max(|k_m D_f - f_{in}|)$ occurs for a frequency located exactly between two bins. This error can be considered as the resolution of the FFT frequency measurement and

$$e = \frac{1}{2}\Delta_f = \frac{f_s}{2N} = \frac{1}{2NT_s} = \frac{1}{2L} \qquad (2)$$

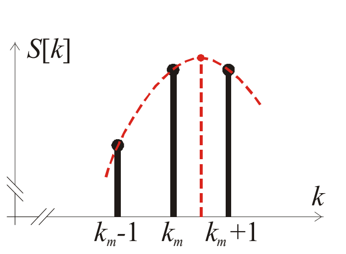
The resolution can be increased considerably by discrete spectrum interpolation, which has been used in tune measurement systems [1]. The method principle is sketched in figures below.



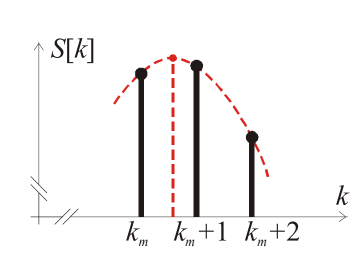
The component f_{in} is located exactly on the largest bin k_m. f_{in} can be calculated from (1) with no error.



When f_{in} increases, the amplitude of bin k_m-1 gets smaller and of bin k_m+1 bigger.



$S[k_m]$ remains the biggest, until f_{in} is equidistant between bins k_m and k_m+1. In this case, determining f_{in} with (1) leads to the largest error (2).



When f_{in} is increased further, bin k_m+1 becomes the biggest. For f_{in} smaller than $k_m D_f$ the analysis is similar resulting in symmetrical cases.

Discrete spectra can be considered as continuous spectrum samples taken at multiples of D_f. Replacing ordinary frequency f by

$$j = \frac{f}{\Delta_f} = f\frac{N}{f_s} = fNT_s = fL \qquad (3)$$

referred to as the normalized frequency, adjusts continuous spectra to the scale of the discrete spectrum bin indexes. It allows expressing discrete and continuous spectra in one convenient scale (e.g. $j=10.5$ for frequency lying exactly between bin 10 and 11).

Results are presented for weighted cosine windows with fast decaying small sidelobes, described by Nuttall [1]

$$w_c(t) = w_r(t)\sum_{i=0}^{3} c_i \cos\left(\frac{2\pi i}{L}t\right) \qquad (4)$$

where $w_r(t)$ is the rectangular window with duration from $-L/2$ to $L/2$. Basic properties of the windows are listed in the **Table**. Magnitude spectra of three of them are plotted in **Fig. 2**.

Three Gaussian windows are also considered. They are defined as

$$w_g(t) = w_r(t)\exp\left(-r^2\frac{t^2}{2L^2}\right) \qquad (5)$$

where r is the ratio between the window length L and its standard deviation s, i.e. the Gaussian shape is truncated at $\pm s\, r/2$.

This paper deals only with spectral interpolation on idealized (i.e. not perturbed by noise nor interference) spectra and in this case only the shape of the main lobe is important, exclusively within the interpolation range of ± 1.5 j units. This part of **Fig. 2** is magnified in **Fig. 3**.

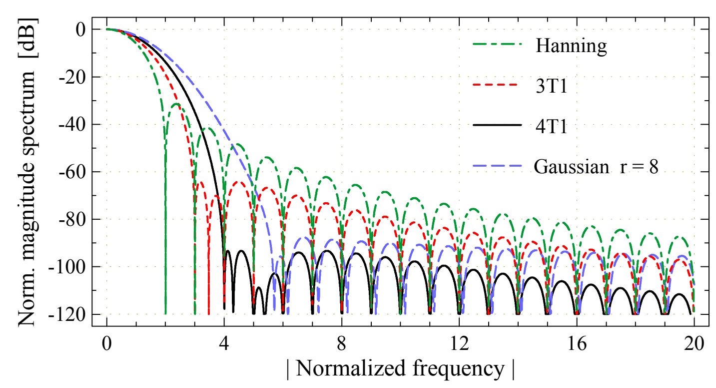


Figure 2. An example of window magnitude spectra.

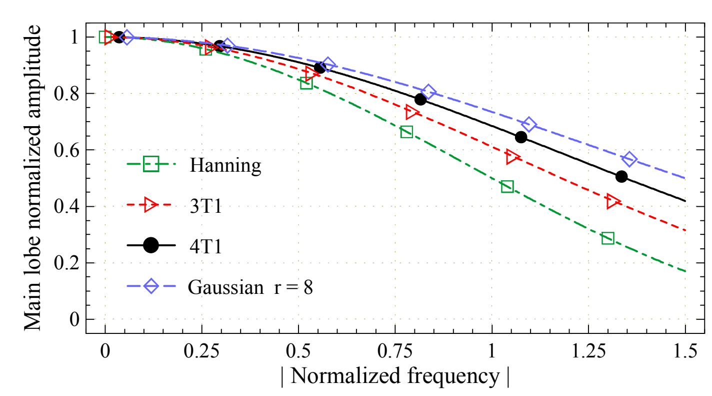


Figure 3. Magnified main lobes of Fig. 2.

Parabolic Interpolation

Let $S[k]$ be the discrete magnitude spectrum of N samples of a signal $s(t)$ containing a sinusoidal component of frequency $j_{in}=f_{in}L$, and k_m be the index of the biggest bin of the corresponding discrete spectrum peak. Fitting a parabola through interpolation nodes $S[k_m-1]$, $S[k_m]$, $S[k_m+1]$ and finding the abscissa of the interpolation maximum j_m, gives

$$j_{in} \cong j_m = k_m + \Delta_f = k_m + \frac{S[k_m+1]-S[k_m-1]}{2\left(2S[k_m]-S[k_m+1]-S[k_m-1]\right)} \qquad (6)$$

The shape of the spectrum peak corresponds to the spectrum of the windowing function applied to the signal samples. If window $w(t)$ with magnitude spectrum $W(j)$ is used, then the interpolation error is [1]

$$E(j_d) = -\frac{W(j_d+1)-W(j_d-1)}{2\left(2W(j_d)-W(j_d+1)-W(j_d-1)\right)} - j_d \qquad (7)$$

where $j_d = j_{in} - k_m$.

The interpolation error corresponding to four windows is shown in **Fig. 4** and is given in units of D_f. For non-perturbed spectra the error is the same around each discrete spectrum bin. The interpolation errors for other windows have similar shapes and can be characterized by the error maximum $E_{max}=\max(|E(j_d)|)$ and its abscissa. They are listed in the **Table**.

Performance of an interpolation method can be characterized by the interpolation gain, defined as the ratio of the FFT frequency resolution (2) and the method maximum error

$$G = \frac{e}{E_{max}} = \frac{\Delta_f}{2E_{max}} \qquad (8)$$

As listed in the **Table**, the parabolic interpolation can increase the resolution of discrete spectra by more than one order of magnitude.

Gaussian Interpolation

The interpolation gain can be significantly improved by fitting a Gaussian shape to find the abscissa of the spectral peak maximum located between two discrete spectrum bins. Since a Gaussian curve is a parabola in the logarithmic scale, the Gaussian interpolation reduces to the parabolic one on the natural logarithm of the magnitude spectrum. Thus, (6) becomes

$$j_{in} \cong j_m = k_m + \frac{\ln S[k_m+1]-\ln S[k_m-1]}{2\left(2\ln S[k_m]-\ln S[k_m+1]-\ln S[k_m-1]\right)} \qquad (9)$$

Similarly, the Gaussian interpolation error can be derived directly from (7). It is plotted in **Fig. 5** for four windows. The errors for other windows have similar shapes and are characterized by the error maximum and its abscissa, as listed in the **Table**. The interpolation gains are about two orders of magnitude for cosine weighted windows and well beyond three orders of magnitude for the Gaussian window of $r=8$.

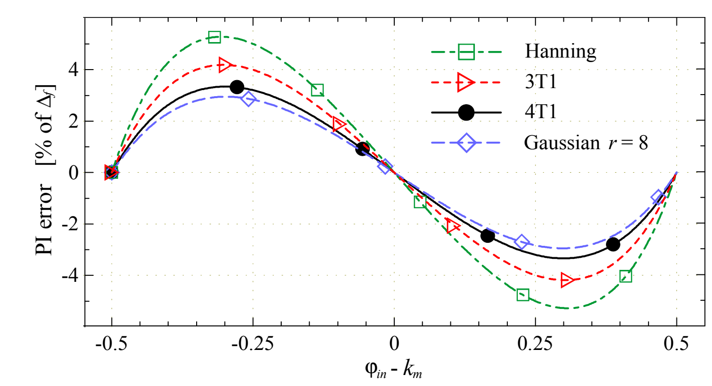


Figure 4. Parabolic interpolation errors.

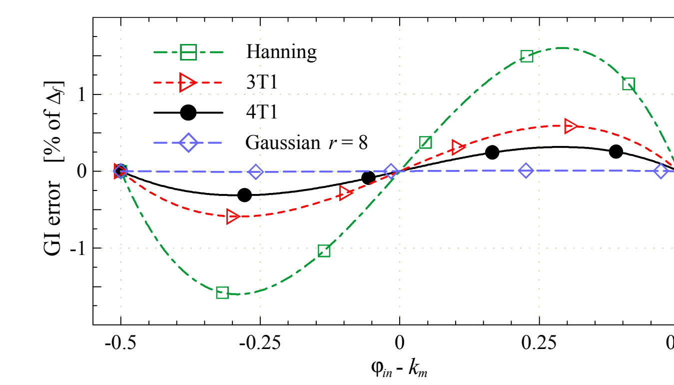
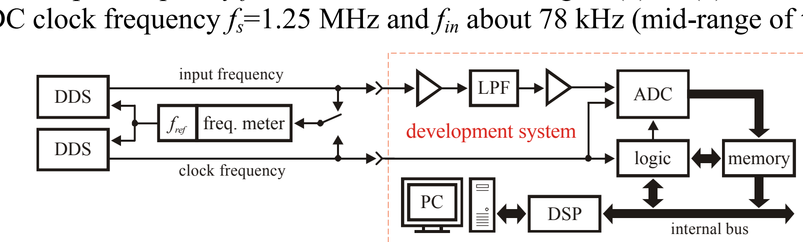


Figure 5. Gaussian interpolation errors.

Measurements

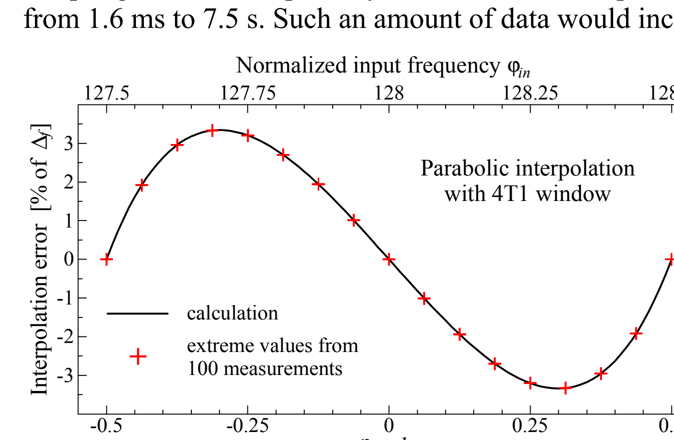
The parabolic and Gaussian interpolation methods were examined with a laboratory setup shown schematically below, based on a tune measurement development system [1]. To achieve input and clock frequencies of sufficient phase stability, the frequencies were generated by two DDSs, driven by the reference source of a frequency meter. During each acquisition, 2048 ADC samples of the sine wave input signal were stored in the memory. Then, the DSP successively performed windowing, the FFT and the power spectrum calculation. Next, the spectrum bin with the biggest amplitude was found and finally the input frequency f_{in} was calculated according to (6) or (9). The measurements were done around bin $k_m=128$, with the ADC clock frequency $f_s=1.25$ MHz and f_{in} about 78 kHz (mid-range of the frequency span of the setup).

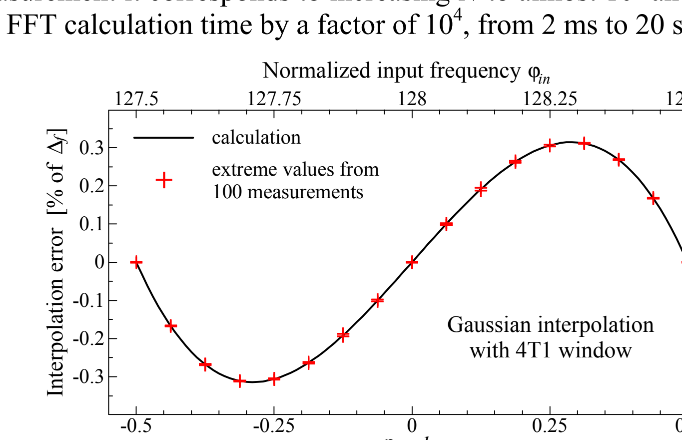


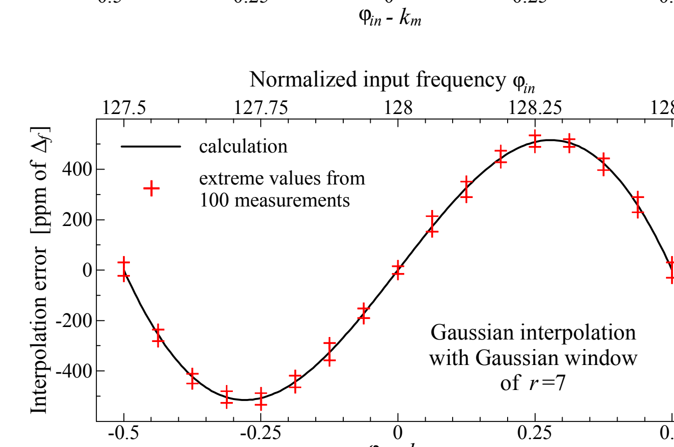
DDS – Direct Digital Synthesizer,
LPF – antialias Low Pass Filter,
ADC – Analog to Digital Converter
DSP – Digital Signal Processor

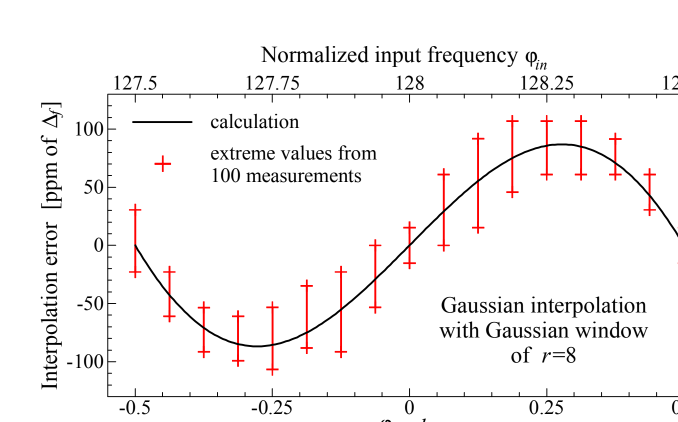
Measurement results are shown in the figures below. Crosses mark extreme values from 100 consecutive measurements with the same setup frequencies and dashed lines show theoretical errors as in **Fig. 4** and **5**. The measurement results were spread due to amplitude noise present in the analyzed spectra, which was converted during the interpolation process into a frequency jitter. This uncertainty was caused mostly by noise present in the input signal, originating in the DDS output 12-bit digital to analog converters. The quantization noise of the ADC might have contributed as well.

For the Gaussian interpolation with Gaussian window of $r=8$ the total of the systematic and "noise" error was 107 ppm of D_f, that is some 65 mHz. The interpolation gain was close to 4700 and relative measurement error of about 0.8 ppm (for f_{in} close to only $f_s/16$). This gain, obtained at the expense of performing the Gaussian interpolation (9) within some microseconds, is equivalent to the frequency resolution of an FFT measurement without interpolation with N samples and the sampling time L multiplied by this factor. For the presented measurement it corresponds to increasing N to almost 10^7 and L from 1.6 ms to 7.5 s. Such an amount of data would increase the FFT calculation time by a factor of 10^4, from 2 ms to 20 s.









Conclusions

Theoretical and experimental studies have been undertaken to enhance FFT frequency measurement resolution, using parabolic or Gaussian interpolations on the discrete magnitude spectrum to find abscissa of spectral peaks maxima located between discrete spectrum bins. The interpolation yield strongly depends upon the windowing function.

In this paper it is shown that the parabolic interpolation can improve the frequency resolution by more than one order of magnitude. Better results can be achieved with Gaussian interpolation. A gain larger than two orders of magnitude is possible with windows having very good spectral properties and well beyond three orders of magnitude when using steep Gaussian windows.

This paper describes systematic errors of the interpolation methods, assuming ideal discrete spectra. Soon, results will be published, concerning the behavior of the methods when spectra are perturbed by noise, interference from strong components and the exponential decay of the input signal.

A direct application of these methods are FFT-based tune measurement systems. The Gaussian method with 4T1 window is used in such a system for the CERN PS Booster accelerator. In the future, similar systems will be made for the PS and LEIR machines.

[1] M. Gasior, J.L. Gonzalez, Improving FFT Frequency Measurement Resolution by Parabolic and Gaussian Interpolation, AB-Note-2004-021 BDI, with all references therein.